A new Einstein-nonlinear electrodynamics solution in 2+1-dimensions

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We obtain a $2+1-$dimensional solution to gravity, coupled with nonlinear electrodynamic Lagrangian of the form of $\sqrt{|F_{\mu\nu}F^{\mu\nu}|}$. The electromagnetic field is considered with an angular component given by $F_{\mu\nu} = E_0\delta^t_\mu\delta^\theta_\nu$ with E_0=constant. We show that the metric coincides with the solution given by Schmidt and Singleton in PLB 721(2013)294 in $2+1-$dimensional gravity coupled with a massless, self interacting real scalar field. Finally the confining motions of massive charged as well as chargeless particles are investigated.

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I. INTRODUCTION

There has already been benefits in studying lower dimensional field theoretical spacetime such as $2+1-$dimensions in general relativity. This is believed to be the projection of higher dimensional cases to the more tractable situations that may inherit physics of intricate higher dimensions. Recent decades proved that the cases of lower dimensions are still far from being easily understandable and in fact entails its own characteristics. The absence of gravitational degree of freedom such as Weyl tensor or pure gravitational waves necessitates endowment of physical sources to fill the blank and create its own curvatures. Among these the most popular addition has been a negative cosmological constant which makes anti-de Sitter spacetimes to the extent that it creates even black holes [1]. Addition of electromagnetic [2] and scalar fields [3, 4] also are potential candidates to be considered in the same context. Beside minimally coupled massless scalar fields which has little significance to add non-minimally self-coupled scalar fields has also been considered. In particular, the real, radial, self-interacting scalar field with a Liouville potential among others seems promising [4]. The distinctive feature of the source in such a study is that the radial pressure turns out to be the only non-zero (i.e., $T^r_r \neq 0$) component of the energy-momentum tensor [4]. In effect such a radial pressure turns out to make a naked singularity but not a black hole. Being motivated by the self-intracting scalar field in $2+1-$dimensional gravity we attempt in this Letter to do similar physics with a non-linear electromagnetic field which naturally has its own non-linearity. Our choice for the Lagrangian in nonlinear electrodynamics (NED) is the square root of the Maxwell invariant, i.e. $(\sqrt{|F_{\mu\nu}F^{\mu\nu}|})$, where as usual the field tensor is defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This Lagrangian belongs to the class of NED with power law Maxwell invariant $|F_{\mu\nu}F^{\mu\nu}|^k$ [5]. Similar Lagrangian in $3+1-$dimensions with magnetic field source has been considered in [6] and with electric field as well as electric and magnetic fields together in [7]. A combination of linear Maxwell with the square root term has been investigated in [8].

Such a Lagrangian naturally breaks the scale invariance, i.e. $x_\mu \to \lambda x_\mu$, $A_\mu \to \frac{1}{\lambda}A_\mu$ for λ=constant, even in $3+1-$dimensions so that interesting results are expected to ensue. Let us add that such a choice of NED has the feature that it doesn't attain the familiar linear Maxwell limit. One physical consequence beside others, of the square-root Maxwell Lagrangian is that it gives rise to confinement for geodesics particles. For a general discussion of confinement in general relativistic field theory the reader may consult [8].

Contrary to previous considerations in this study [4] our electric field is not radial, instead our field tensor is expressed in the form $F_{\mu\nu} = E_0\delta^t_\mu\delta^\theta_\nu$ for E_0 =constant. This amounts to the choice for the electromagnetic vector potential $A_\mu = E_0\,(a\theta, 0, bt)$, where our spacetime coordinates are labelled as $x^\mu = \{t, r, \theta\}$ and the constants a and b satisfy $a+b=1$. The particular choice $a=0$, $b=1$ leaves us with the vector potential $A_\mu = \delta^\theta_\mu E_0 t$, which yields a uniform field in the angular direction. The only non vanishing energy momentum tensor component is T^r_r which accounts for the radial pressure. Our ansatz electromagnetic field in the circularly symmetric static metric gives a solution that is identical with the spacetime obtained from an entirely different sources, namely the self-intracting real scalar fields. This is a conformally flat anti-de Sitter solution in $2+1-$dimensions without formation of a black hole. The uniform electric field self-intracting is strong enough to make a naked singularity at the circular center.

In analogy with the $3+1-$dimensions [8] we search for possible particle confinement in this $2+1-$dimensional model with $\Lambda = 0$. Truly the geodesics for both neutral and charged particles are confined.

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II. FIELD EQUATIONS AND THE SOLUTION

We start with the following action for the Einstein's theory of gravity coupled with a NED Lagrangian

$$I = \frac{1}{2} \int dx^3 \sqrt{-g} \left(R - 2\Lambda + \alpha \sqrt{|\mathcal{F}|} \right). \tag{1}$$

Here $\mathcal{F} = F_{\mu\nu} F^{\mu\nu}$ is the Maxwell invariant with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, α is a real coupling constant and Λ is the cosmological constant. Our line element is circularly symmetric given by

$$ds^2 = -A(r)\, dt^2 + B(r)\, dr^2 + R(r)\, d\theta^2, \tag{2}$$

where $A(r), B(r)$ and $R(r)$ are three unknown functions of r and $0 \leq \theta \leq 2\pi$. Also we choose the field ansatz as

$$\mathbf{F} = E_0 dt \wedge d\theta \tag{3}$$

in which E_0=constant, is a uniform electric field and its dual can be found as ${}^\star\mathbf{F} = E_0 \sqrt{\frac{B(r)}{A(r)R(r)}} dr$. This electric field derives from an electric potential one-form given by

$$\mathbf{A} = E_0 \left(a t d\theta - b \theta dt \right) \tag{4}$$

in which a and b are constants satisfying $a + b = 1$. The nonlinear Maxwell's equation reads

$$d \left({}^\star\mathbf{F} \frac{\sqrt{|\mathcal{F}|}}{\mathcal{F}} \right) = 0, \tag{5}$$

which upon substitution

$$\mathcal{F} = 2F_{t\theta} F^{t\theta} = \frac{-2E_0^2}{A(r)\, R(r)} \tag{6}$$

is trivially satisfied. Next, the Einstein-NED equations are given by

$$G_\mu^\nu + \Lambda g_\mu^\nu = T_\mu^\nu \tag{7}$$

in which

$$T^\mu{}_\nu = \frac{\alpha}{2} \sqrt{|\mathcal{F}|} \left(\delta^\mu{}_\nu - \frac{2\left(F_{\nu\lambda} F^{\mu\lambda} \right)}{\mathcal{F}} \right). \tag{8}$$

Having $\mathcal{F}$ known one finds

$$T^t{}_t = T^\theta{}_\theta = 0, \tag{9}$$

while

$$T^r{}_r = \frac{\alpha}{2} \sqrt{\frac{2E_0^2}{A(r)\, R(r)}} \tag{10}$$

as the only non-vanishing energy-momentum component. We note that in our calculation, implicitly, we assumed $A(r)\, R(r) \geq 0$. To proceed further, we must have the exact form of the Einstein tensor components which are given by

$$G_t^t = \frac{2R''BR - R'B'R - R'^2 B}{4B^2 R^2} \tag{11}$$

$$G_r^r = \frac{A'R'}{4ARB} \tag{12}$$

and

$$G_{\theta}^{\theta} = -\frac{A'B'A - 2A''BA + A'^2B}{4B^2A^2}. \tag{13}$$

The field equations then read

$$\frac{2R''BR - R'B'R - R'^2B}{4B^2R^2} + \Lambda = 0, \tag{14}$$

$$\frac{A'R'}{4ARB} + \Lambda = \frac{\alpha}{2}\sqrt{\frac{2E_0^2}{A(r)\,R(r)}} \tag{15}$$

and

$$\frac{2A''BA - A'B'A - A'^2B}{4B^2A^2} + \Lambda = 0. \tag{16}$$

From the first and the last field equations it is clear that there is a symmetry between $A(r)$ and $R(r)$ which suggests they are proportional, i.e. $A(r) = \xi R(r)$ where ξ is a constant which upon redefinition of time we set it to unity. Considering these results together reduces the field equations into the following equations

$$\frac{A'^2}{AB} + 4\Lambda = \frac{4\alpha E_0}{\sqrt{2}} \tag{17}$$

and

$$\left(\frac{A'^2}{BA}\right)' + 4\Lambda A' = 0. \tag{18}$$

The second equation, using the first, effectively becomes

$$4\Lambda A' = 0. \tag{19}$$

Here we have two different cases:

A. $\Lambda \neq 0$

For the case of non-zero cosmological constant, Eq. (17) yields $A' = 0$ and consequently $A = R = const.$ while $B(r)$ is arbitrary. Having no $r-$dependence in g_{tt} and $g_{\theta\theta}$ one may introduce $dT^2 = Adt^2$, $dX^2 = Bdr^2$ and $dY^2 = Rd\theta^2$so that the effective line element will locally be

$$ds^2 = -dT^2 + dX^2 + dY^2, \tag{20}$$

which is flat and may be of little interest. What happens here in effect is that the constant electric field E_0 and cosmological constant cancel each other mutually to result in a $2+1-$dimensional flat spacetime.

B. $\Lambda = 0$

In the case of zero cosmological constant, Eq. (19) is satisfied and the only equation remained is Eq. (18) with $\Lambda = 0$. This equation admits

$$B = \frac{A'^2}{2\sqrt{2}\alpha E_0 A} \tag{21}$$

which is a relation between the unknown functions A and B. After $A = R$ and the latter constrain, by virtue of the following transformation

$$\bar{r} = \int \sqrt{AB}dr \tag{22}$$

the line element (2) can be cast into

$$ds^2 = -f(\bar{r})\,dt^2 + \frac{d\bar{r}^2}{f(\bar{r})} + f(\bar{r})\,d\theta^2, \tag{23}$$

in which $f(\bar{r}) = A(r(\bar{r}))$ .This means that without lose of generality one can consider $AB = 1$ which leads to

$$A' = \sqrt{2\sqrt{2}\alpha E_0} \tag{24}$$

which amounts to

$$A = \xi r + C \tag{25}$$

where C is an integration constant and $\xi = \sqrt{2\sqrt{2}\alpha E_0}$. Finally the line element becomes

$$ds^2 = -(\xi r + C)\,dt^2 + \frac{dr^2}{(\xi r + C)} + (\xi r + C)\,d\theta^2. \tag{26}$$

We comment here that this solution is not a black hole solution due to $-g_{tt} = g_{\theta\theta}$ and therefore by a simple shift of the origin i.e. $r \to r - \frac{C}{\xi}$ one can set C to zero. This therefore, implies

$$ds^2 = -\xi r dt^2 + \frac{dr^2}{\xi r} + \xi r d\theta^2. \tag{27}$$

To compare our solution with the known solution given in [4] we apply the following transformation

$$r = \frac{\xi}{4}\tilde{r}^2, \theta = \frac{\tilde{\theta}}{\xi}, t = \frac{\tilde{t}}{\xi} \tag{28}$$

which casts the line element into

$$ds^2 = -\tilde{r}^2 d\tilde{t}^2 + d\tilde{r}^2 + \tilde{r}^2 d\tilde{\theta}^2. \tag{29}$$

We notice that although our solution is not a standard black hole there exists still a horizon at $r = 0$ which makes our solution a black point [9, 10]. In [9] such black points appeared in $3+1-$dimensional gravity coupled to the logarithmic $U(1)$ gauge theory and in [10] coupled to charged dilatonic fields. This is the conformally flat $2+1-$dimensional line element through the transformation $\tilde{r} = e^R$, given by

$$ds^2 = e^{2R}\left(-d\tilde{t}^2 + dR^2 + d\tilde{\theta}^2\right) \tag{30}$$

obtained also in the self-intracting scalar field model [4].

III. GEODESIC MOTION

A. Chargeless Particle

To identify more the solution found above one may study the geodesic motion of a massive particle (time-like). The Lagrangian of the motion of a unit mass particle within the spacetime (29) is given by (for simplicity we use r instead of $\tilde{r}$)

$$L = -\frac{1}{2}r^2\dot{t}^2 + \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 \tag{31}$$

where a 'dot' denotes derivative $\frac{d}{ds}$ with s an affine parameter. The conserved quantities are

$$\frac{\partial L}{\partial \dot{t}} = -r^2\dot{t} = -\alpha_0 \tag{32}$$

$$\frac{\partial L}{\partial \dot{\theta}} = r^2\dot{\theta} = \beta_0 \tag{33}$$

with α_0 and β_0 as constants of energy and angular momentum. The time like metric condition reads

$$-1 = -r^2\dot{t}^2 + \dot{r}^2 + r^2\dot{\theta}^2 \tag{34}$$

which upon using (32) and (33) yields

$$\dot{r}^2 = \left(\frac{\alpha_0^2 - \beta_0^2}{r^2} - 1\right). \tag{35}$$

This equation clearly shows a confinement in the motion for the particle geodesics in the form

$$r^2 \leq \alpha_0^2 - \beta_0^2. \tag{36}$$

Considering the affine parameter as the proper distance one finds from (35),

$$r = \sqrt{\alpha_0^2 - \beta_0^2 - (s - s_0)^2}. \tag{37}$$

B. Charged Particle Geodesics

For a massive charged particle with unit mass and charge q_0 the Lagrangian is given by

$$L = -\frac{1}{2}r^2\dot{t}^2 + \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 + q_0 A_\mu \dot{x}^\mu \tag{38}$$

in which $A_\mu \dot{x}^\mu = A_\theta \dot{x}^\theta = E_0 t\dot{\theta}$ i.e. the choice $a = 1$, $b = 0$ in Eq. (4). The metric condition is as (34) and therefore the Lagrange equations yield

$$\frac{d}{ds}\left(r^2\dot{\theta} + q_0 E_0 t\right) = 0, \tag{39}$$

$$\frac{d}{ds}\left(r^2\dot{t}\right) = -q_0 E_0 \dot{\theta} \tag{40}$$

and

$$\ddot{r} = -r\dot{t}^2 + r\dot{\theta}^2. \tag{41}$$

The first equation implies

$$r^2\dot{\theta} + q_0 E_0 t = \beta_0 = const. \tag{42}$$

while the second equation with a change of variable as $r^2\frac{d}{ds} = \frac{d}{dz}$ and imposing (42) yields

$$\frac{d^2 t}{dz^2} = -q_0 E_0 \left(\beta_0 - q_0 E_0 t\right). \tag{43}$$

This equation admits an exact solution for $t(z)$

$$t(z) = \frac{\beta_0}{\omega} + C_1 e^{\omega z} + C_2 e^{-\omega z} \tag{44}$$

in which C_1 and C_2 are two integration constants and $\omega = q_0 E_0$. Next, the radial equation upon imposing the metric condition (34) becomes decoupled as

$$r\ddot{r} + \dot{r}^2 + 1 = 0. \tag{45}$$

The general solution for this equation is given by

$$r = \pm\sqrt{\tilde{C}_1 + 2\tilde{C}_2 s - s^2} \tag{46}$$

where we consider the positive root. Once more going back to Eq. (44) one may write

$$r^2 \frac{dt}{ds} = \frac{dt}{dz} \tag{47}$$

which in turn becomes

$$\left(\tilde{C}_1 + 2\tilde{C}_2 s - s^2\right) \frac{dt}{ds} = \omega \left(C_1 e^{\omega z} - C_2 e^{-\omega z}\right). \tag{48}$$

To proceed further we set $\tilde{C}_2 = 0, \tilde{C}_1 = b_0^2$, $C_2 = 0$ and $C_1 = 1$ so that (one must note that with this choice of integration constants $b_0^2 - s^2 \geq 0$)

$$\left(b_0^2 - s^2\right) \frac{dt}{ds} = \omega \left(t - \frac{\beta_0}{\omega}\right) \tag{49}$$

which finally leads to

$$\left| t - \frac{\beta_0}{\omega} \right| = \left(\frac{b_0 + s}{b_0 - s}\right)^{\frac{\omega}{2b_0}} + \zeta, \tag{50}$$

in which ζ is an integration constant which we set it to zero. Having latter relation one can find

$$r(t) = \frac{2b_0 \left| t - \frac{\beta_0}{\omega} \right|^{\frac{b_0}{\omega}}}{\left| t - \frac{\beta_0}{\omega} \right|^{\frac{2b_0}{\omega}} + 1} \tag{51}$$

which clearly shows a confining motion for the charged particle. Such conclusion could also be manifested by Eq. (46) which implies $\tilde{C}_1 + 2\tilde{C}_2 s - s^2 \geq 0$ and consequently $r \leq \sqrt{\tilde{C}_1 + \tilde{C}_2^2}$. The angular variable $\theta(t)$ can also be reduced to an integral expression.

IV. CONCLUSION

We considered a specific form of NED Lagrangian in the form of power law Maxwell invariant $|F_{\mu\nu}F^{\mu\nu}|^k$ with $k = \frac{1}{2}$. It is known that a pure radial electric field with $k = \frac{1}{2}$ is not feasible in connection with the energy conditions [11] so our field ansatz has been chosen to be a uniform angular electric field. One direct feature of this form of field ansatz appeared in the form of the energy momentum tensor whose only non-zero component is found to be T_r^r. This indeed means that the energy density $\rho = -T_t^t$ and the angular pressure $p_\theta = T_\theta^\theta$ are zero while the radial pressure $p_r = T_r^r = \frac{\xi}{r}$ where $\xi = \sqrt{\frac{\sqrt{2}\alpha E_0}{8}}$. These make the weak energy conditions to be satisfied, i.e., $\rho \geq 0$, $\rho + p_r \geq 0$ and $\rho + p_\theta \geq 0$. Even further the strong energy conditions are also satisfied which are WECs together with $\rho + p_r + p_\theta \geq 0$. Having radial pressure non-zero and divergence at $r = 0$ has additional feature which can be seen from the nature of the spacetime which although it is not a black hole solution but is singular at $r = 0$ such that the spacetime's invariants are $R \sim \frac{1}{r^2}$, $R_{\mu\nu}R^{\mu\nu} \sim \frac{1}{r^4}$ and Kretschmann scalar$\sim \frac{1}{r^4}$ (using (29)). This singularity is in the same order of divergence as the charged BTZ black hole with radial singular electric field. As it has been found in this work, the singularity at the origin ($r = 0$ is also a zero for g_{tt} which makes our solution a black point) confines the radial motion of a massive particle (charged or chargeless). This means that the particle can not go beyond a maximum radius (see Eq.s (37) and (46)). As our final remark we would like to refer to the work of Schmidt and Singleton [4] where by virtue of a different matter source, namely a self-interacting scalar field, the same solution has been found. This is perhaps an indication that the black hole solution found in the context of a real scalar field sharing common metric with different sources in $2+1-$dimensions may apply to higher dimensional spacetimes.

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